The College Of WILLIAM & MARY

Math 112
Lab 2

Volumes

I. Rotating a solid about the *x*-axis

In section 6.2, we learn how to calculate the volume of a solid created by rotating a plane about the *x*-axis. That is, if the solid lies between $x = a$ and $x = b$ and all cross-sectional areas are created by making vertical "cuts", perpendicular to the *x*-axis, then the volume of the solid is the limit of Riemann sums.

$$V = \lim_{n\to\infty}\sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)dx \qquad 2.1$$

Since the plane is rotated about an axis, the cross-sectional areas are all circles.

If we want to *approximate* the volume, we fix the number of subintervals, *n*, rather than allowing *n* to go to infinity. That is,

$$V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x \qquad 2.2$$

(WORK EXERCISE 1)

When the cross-sectional planes have the shape of washers, rather than disks, the area must exclude the hole in the middle. Given a representative washer with outside radius, x_1, and inside radius, x_2, the area is

$$A(x) = \pi x_1^2 - \pi x_2^2 = \pi\left(x_1^2 - x_2^2\right) \qquad 2.3$$

(WORK EXERCISE 2)

II. Rotating a solid about the *y*-axis

When a solid is created by rotating a plane about the *y*-axis, the cross-sectional areas are created making horizontal "cuts", perpendicular to the *y*-axis. If the solid lies between $y = c$ and $y = d$, then the volume of the solid is the limit of Riemann sums

$$V = \lim_{n\to\infty}\sum_{i=1}^{n} A(y_i^*)\Delta y = \int_a^b A(y)dy \qquad 2.4$$

Again, the plane is rotated about an axis so the cross-sectional areas are all circles. Notice the area formula, disk thickness Δy and integration are with respect to *y*.

If we want to *approximate* the volume, we fix the number of subintervals, *n*, rather than allowing *n* to go to infinity. That is,

$$V \approx \sum_{i=1}^{n} A(y_i^*)\Delta y \qquad 2.5$$

Once again, if the cross-sectional planes have the shape of washers, rather than disks, the area must exclude the hole in the middle.

$$A(y) = \pi y_1^2 - \pi y_2^2 = \pi\left(y_1^2 - y_2^2\right) \qquad 2.6$$

(WORK EXERCISE 3)

III. Solids of revolution about any horizontal or vertical line

When a solid is created by rotating a plane about $y = a$ or $x = a$ where $a \neq 0$ (not the x- or y-axis), use a sketch of the solid to derive the formula for the radius of a representative disk or radii if the cross-sectional planes are washers.

For example, consider the solid obtained when the region bound by $y = \sqrt{3x}$ and $y = x$ is rotated about the line $x = 4$. The cross-sectional planes are washers, perpendicular to the line $x = 4$. Express x in terms of y for each of the boundary curves then use the sketch to derive the formulas for the inside and outside radii. We see the inside radius is the distance from the line $x = 4$ to the curve $x = y$. For any representative washer between $y = 0$ and $y = 3$, the inside radius is equal to $4 - y$. We also see the outside radius for any representative washer is the distance from the line $y = 4$ to the curve $x = \frac{1}{3}y^2$ so the outside radius is equal to $4 - \frac{1}{3}y^2$. The equation for the area of any cross-sectional washer is:

$$A(y) = \pi\left[\left(4 - \frac{1}{3}y^2\right)^2 - (4 - y)^2\right]$$

The volume of the resulting solid is calculated by solving the integral

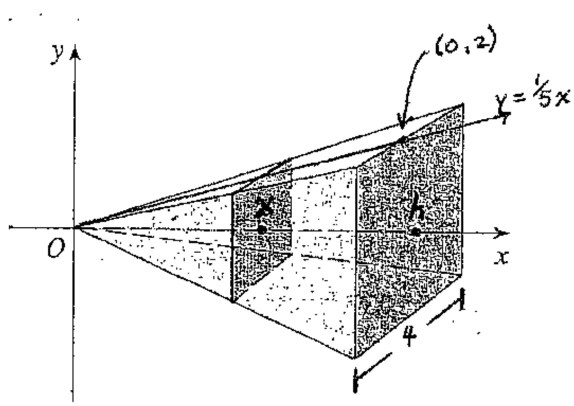
$$V = \int_0^3 \pi\left[\left(4 - \frac{1}{3}y^2\right)^2 - (4 - y)^2\right]dy$$

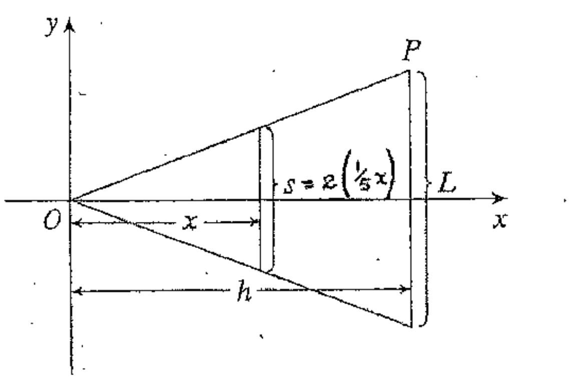
(WORK EXERCISE 4)

IV. Solids that are not solids of revolution

Integration may still be used to find the volume of solids not obtained by revolving a plane about an axis. If each cross-sectional plane has the same geometric shape for which we have an area formula, the proper integral may be established.

Consider a pyramid whose base is a square with sides of 4 inches and whose height is 10 inches. To find the volume of this pyramid (using calculus), orient the pyramid such that the x-axis runs from the apex at (0, 0) and (10, 0) is at the center of the square base. Notice each cross-sectional cut is a square. Although we know a formula for the area of a square, we need a formula for the length of the sides at any representative cross-sectional plane. The points (0, 0) and (10, 2) lie on a line along the side of the pyramid above the *x*-axis. Using these points, the equation of the line is $y = \frac{1}{5}x$. So, for any *x*-value between 0 and 10, the square has sides equal to $2 \times \frac{1}{5}x$. The volume of the pyramid is calculated by solving the integral $\int_0^{10} \left(\frac{2}{5}x\right)^2 dx$.





(WORK EXERCISE 5)

Math 112 F24 Lab 2 Exercises Name: ____________________________ Section: ______ Score: _____

You may use your textbook, lab and notes. Students may work cooperatively but must submit their own set of Lab Exercises. No calculators unless noted.

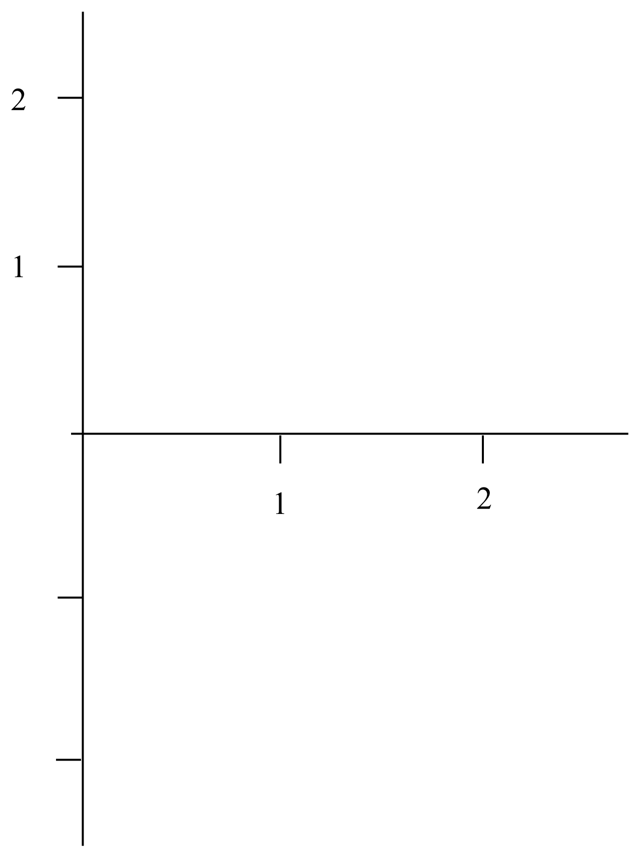
1. (a) Explain the meaning of $A(x_i^*)$ and Δx in the approximation $V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x$.

(b) Sketch the solid obtained by rotating about the x-axis, the region bound by $y = 1 + \dfrac{x^2}{4}$, $x = 0$ and $x = 2$. Using $n = 4$ and midpoints, complete $i - iii$ below then set up and calculate the Riemann sum to approximate the volume of the solid (use a calculator and maintain three decimal places throughout your calculations.)

i)What is the value of Δx ?: ____

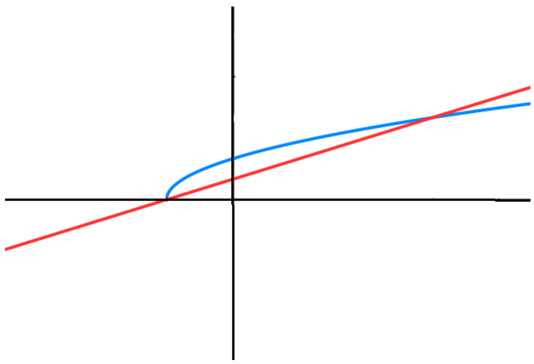
ii)List the midpoints used to find $A(x_i)$: ____

iii)Write the formula for $A(x_i^*)$: ____



(c) Illustrate this midpoint approximation by sketching above, the four subintervals as well as the disks created. The disks will appear as rectangles on your two-dimensional sketch. (A three-dimensional example is on page 439 in the textbook.)

2.(a) Set up and solve the proper equation to find the intersection(s) of the two curves $y=\sqrt{x+1}$ and $y=\frac{1}{2}(x+1)$. (show your work)



(b) On the graph above, sketch the solid obtained by rotating the bounded region about the line $y=-1$.

(c) Write the expression for the area ($A(x)$ or $A(y)$) of a representative washer.

(d) Set up and solve the definite integral used to calculate the volume of the solid in (c). Your work should include the proper antiderivative. Express your answer as a fraction containing π, not a decimal value.

3. The base of a solid is the region bound by $y = \sin(x)$ and the x-axis from $x = 0$ to $x = \pi$. Each cross-sectional cut, perpendicular to the x-axis, is an equilateral triangle sitting on this base.

(a) Find the equation for $A(x)$, the area of each cross-sectional cut.

(b) Set up and solve the definite integral for volume of this solid. (Hint: you will need the identity $\sin^2(x) = \frac{1}{2}[1 - \cos(2x)]$ to evaluate the integral.)