Representing Functions as Power Series

I. Introduction

In section 11.8 we learned the series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots \tag{1}$$

is called a power series. It is a function of x whose domain is the set of all x for which it converges.

In section 11.2 we learned the series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots + ar^n + \ldots \qquad a \neq 0, \tag{2}$$

is a geometric series that converges to the sum, $s = \dfrac{a}{1-r}$, if $|r| < 1$.

If we let $c_n = 1$ in (1), the power series becomes the geometric series where $a = 1$ and $r = x$. It follows, if $|x| < 1$, the power series converges to the sum, $s = \dfrac{1}{1-x}$. Thus, we see a function that can be represented as a power series:

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots \text{ whose domain is } |x| < 1 \tag{3}$$

II. Creating Power Series Representations for Other Functions

By manipulating the expressions in equation (3), we can represent other functions as powers series. This manipulation includes algebra, substitution, differentiation and integration. The following three examples demonstrate how this works. Simpler examples are in your textbook on pages 747-750.

Example 1: Express $f(x) = \dfrac{x}{2x^2+3}$ as a power series.

At first glance, this shows no resemblance to the function in (3) but we use algebra to manipulate $f(x)$ and create a substitution for x in the geometric series $\sum_{n=0}^{\infty} x^n$. Generally, we want the form:

$f(x) = h(x)\cdot\dfrac{1}{1-g(x)} = \sum_{n=0}^{\infty}[g(x)]^n \cdot h(x)$. For the function above we factor out x from the numerator and 3 from the denominator. Then, replace x in equation (3) with $-\dfrac{2}{3}x^2$. So,

$$f(x) = \frac{x}{2x^2+3} = \frac{x}{3}\cdot\frac{1}{1-\left(-\frac{2}{3}x^2\right)} = \frac{x}{3}\cdot\sum_{n=0}^{\infty}\left(-\frac{2}{3}x^2\right)^n = \sum_{n=0}^{\infty}\left(-\frac{2}{3}x^2\right)^n\cdot\frac{x}{3} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{3}\left(\frac{2}{3}\right)^n x^{2n+1}$$

The power series representation for $f(x) = \dfrac{x}{2x^2+3}$ is $\sum_{n=0}^{\infty}(-1)^n\dfrac{1}{3}\left(\dfrac{2}{3}\right)^n x^{2n+1}$.

Rules for differentiation and integration of functions can be applied in power series problems.

A. Using the quotient rule for differentiation we see

$$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{(1-x)\frac{d}{dx}(1)-(1)\frac{d}{dx}(1-x)}{(1-x)^2} = \frac{1}{(1-x)^2}$$

We can also differentiate a power series term-by-term to obtain:

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty}x^n\right] = \frac{d}{dx}\left[1+x+x^2+x^3+\ldots\right] = 0+1+2x+3x^2+\ldots = \sum_{n=1}^{\infty}nx^{n-1}$$

B. Using the indefinite integral $\int\frac{1}{x}dx = \ln|x|+C$ along with substitution we see

$$\int\frac{1}{1-x}dx = -\ln|1-x|+C$$

We can also integrate a power series term-by-term to obtain:

$$-\ln(1-x) = \int\frac{1}{1-x}dx = \int\sum_{n=0}^{\infty}x^n dx = \int\left[1+x+x^2+x^3+\ldots\right]dx = x+\frac{1}{2}x^2+\frac{1}{3}x^3+\ldots+C = \sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1}$$

Example 2: Find the power series representation for $f(x)=\dfrac{x}{(2x+3)^2}$.

Notice the difference between this function and the one in Example 1. The entire denominator is squared here which provides a "clue" as to what type of manipulation is needed. The quotient rule for differentiation results in squaring the denominator so we know the function we must differentiate has ($2x$+3) in the denominator. The x in the numerator can be factored out and "ignored" temporarily. We see manipulating $f(x)$ requires a little investigating but a good starting point is to differentiate 1/(2x+3):

$$\frac{d}{dx}\left[\frac{1}{2x+3}\right]=\frac{(2x+3)(0)-1(2)}{(2x+3)^2}=\frac{-2}{(2x+3)^2}$$

Now that we know what the derivative looks like, we can write $f(x)$ in terms of this derivative.

$$f(x)=\frac{x}{(2x+3)^2}=\frac{x}{-2}\cdot\frac{-2}{(2x+3)^2}=-\frac{x}{2}\frac{d}{dx}\left[\frac{1}{2x+3}\right]$$

Why do we go through all these manipulations? We can now create a power series for 1/(2x+3) and differentiate it to create another power series. We then include the factor $-\dfrac{x}{2}$ in the power series.

$$f(x)=-\frac{x}{2}\frac{d}{dx}\left[\frac{1}{2x+3}\right]=-\frac{x}{2}\frac{d}{dx}\left[\frac{1}{3}\cdot\frac{1}{1-\left(-\frac{2}{3}x\right)}\right]=-\frac{x}{6}\frac{d}{dx}\left[\sum_{n=0}^{\infty}\left(-\frac{2}{3}x\right)^n\right]=-\frac{x}{6}\frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{3}\right)^n x^n\right]$$

$$=-\frac{x}{6}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{3}\right)^n nx^{n-1}=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{6}\left(\frac{2}{3}\right)^n nx^n$$

The power series representation for $f(x)=\dfrac{x}{(2x+3)^2}$ is $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{6}\left(\frac{2}{3}\right)^n nx^n$.

There are a few important points worth noting.

1. The derivative of a power series IS another power series! The same is true for the integral of a power series.

2. The radius of convergence remains the same when a power series is differentiated or integrated but the interval of convergence might change.

Example 3: Find the power series representation for $f(x) = \ln(2x+3)$.

Once again, there is a connection between 1/(2x+3) and the given function. Integration provides this connection so a good starting point in creating the power series for $f(x)$ is to integrate 1/(2x+3).

$$\int \frac{1}{2x+3}\,dx = \frac{1}{2}\ln|2x+3| + C$$

Now that we see the integral only differs from $f(x)$ by a factor of 1/2, we can write $f(x)$ in terms of this integral.

$$f(x) = \ln(2x+3) = 2\cdot\frac{1}{2}\ln|2x+3| = 2\int\frac{1}{2x+3}\,dx$$

We can now create a power series for 1/(2x+3) and integrate it to create the power series for $f(x)$. We then include the factor 2 in the power series.

$$f(x) = 2\int\frac{1}{2x+3}\,dx = 2\int\frac{1}{3}\cdot\frac{1}{1-\left(-\frac{2}{3}x\right)}\,dx = \frac{2}{3}\int\sum_{n=0}^{\infty}\left(-\frac{2}{3}x\right)^n dx = \frac{2}{3}\int\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{3}\right)^n x^n\,dx$$

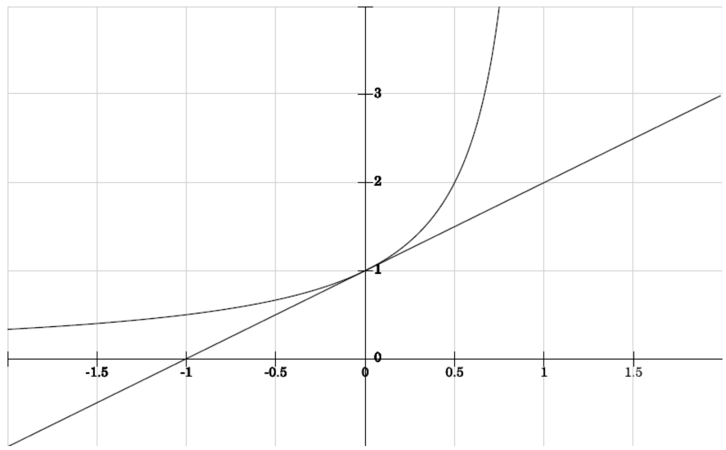
$$= \frac{2}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{3}\right)^n\frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{3}\right)^{n+1}\frac{x^{n+1}}{n+1}$$

The power series representation for $f(x) = \ln(2x+3)$ is $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2}{3}\right)^n\frac{x^n}{n}$.

III. Approximating a Function using a Power Series

We learned in 11.3 that any partial sum s_n can approximate the sum of a convergent series. How does this relate to our power series representations of a function? Suppose we want to approximate the sum of the convergent geometric power series $\sum_{n=0}^{\infty}x^n = 1 + x + x^2 + x^3 + \ldots$ at $x = \frac{1}{4}$ (remember, this series only converges for $-1 < x < 1$). Using the 2nd partial sum, we get $s \approx s_2 = 1 + \frac{1}{4} = \frac{5}{4}$. We know the exact sum is $s = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$ since $\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n = 1 + x + x^2 + x^3 + \ldots$ Calculating s_2 amounts to summing the first two terms in the series. So, if we do not fix the value of x, the approximation of the sum is a linear equation in x. That is $\frac{1}{1-x} \approx 1 + x$ for $|x| < 1$.

Below are the graphs of $f(x)=\dfrac{1}{1-x}$ and the linear approximation $f(x)=1+x$. Notice, the line provides a good approximation of the function for values of x that are very close to $x = 0$.



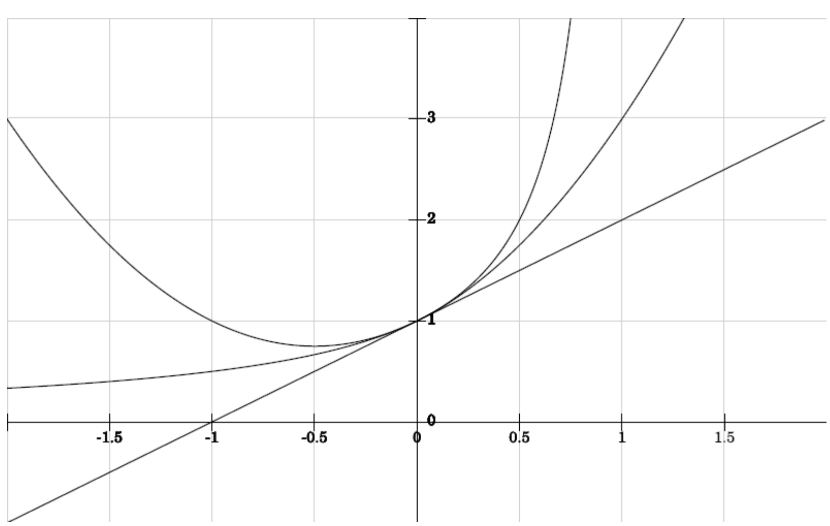
In Math 111, we learned a formula for the linear approximation of a function near a point $x = a$.

Given a function $f(x)$, if $x \approx a$ then $$f(x) \approx f(a) + f'(a)(x-a) \qquad (4)$$

Using our function $f(x)=\dfrac{1}{1-x}$ and $x = 0$, we use this formula and get the same linear approximation we obtained from the power series. This leads to a few questions.

Question 1. Since s_3 provides a better approximation than s_2 for the sum of a convergent series, does adding another term in the power series also show a better approximation (graphically)?

Adding the next term of the power series for an approximation yields $\dfrac{1}{1-x} \approx 1+x+x^2$. We can see this quadratic approximation is better.



Question 2. Is there a connection to the linear approximation formula in the box above and the power series representation of a function? If so, does this connection extend to the quadratic approximation as well?

The answer is "yes" and we will investigate.

The *linear* approximation formula $\frac{1}{1-x} \approx 1 + x$, satisfies the following conditions:

- Condition 1: At $x = a = 0$, $f(x)$ and the tangent line meet.
- Condition 2: At $x = a = 0$, $f(x)$ and the tangent line have the same slope (*i.e.* their derivatives are equal at $x = 0$.)

The general formula that satisfies these conditions (for $a = 0$) is $f(x) = f(0) + f'(0)x$ (5)

The *quadratic* approximation formula $\frac{1}{1-x} \approx 1 + x + x^2$, satisfies the following conditions:

- Condition 1: At $x = a = 0$, $f(x)$ and the tangent parabola meet.
- Condition 2: At $x = a = 0$, $f(x)$ and the tangent parabola have the same slope (*i.e.* their derivatives are equal at $x = 0$.)
- Condition 3: At $x = a = 0$, $f(x)$ and the tangent parabola have the same concavity (*i.e.* their second derivatives are equal at $x = 0$.)

The general formula that satisfies these conditions (for $a = 0$) is $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$ (6)

Question 3. Adding more terms to s_n yields a better approximation of the sum; does adding more terms in the power series yield a better approximation to the function $f(x)$? Does the general formula for the approximating polynomial continue to have a pattern? The answers are "yes" and "yes".

Each higher degree polynomial that approximates $f(x)$ will satisfy the previous conditions analogous to the tangent line and the tangent parabola as well as:

- The n$^{\text{th}}$ derivatives of $f(x)$ and the tangent n$^{\text{th}}$ degree polynomial are equal at $x = 0$.

The general formula that satisfies these conditions (for $a = 0$) is

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \qquad (7)$$

This series is called the **Maclaurin Series**. Notice the formula allows us to find power series representations for other functions.

Math 112 F24 Lab 10 Exercises Name: ________________________Section: ____Score: ____

You may use your textbook, lab and notes. Students may work cooperatively but must submit their own set of Lab Exercises. No calculators unless noted.

1. (a) In Example 2 why does the series beginning at $n = 0$ change to $n = 1$, as shown?

$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{6}\left(\frac{2}{3}\right)^{n}nx^{n}=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{6}\left(\frac{2}{3}\right)^{n}nx^{n}$$

(b) In Example 3 why does the series beginning at $n = 0$ change to $n = 1$, as shown?

$$\sum_{n=0}^{\infty}(-1)^{n}\left(\frac{2}{3}\right)^{n+1}\frac{x^{n+1}}{n+1}=\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2}{3}\right)^{n}\frac{x^{n}}{n}$$

(c) It is said (in class, in the textbook and in this lab) that $\frac{1}{1-x}=1+x+x^2+x^3+\ldots$ but there seems to be a problem. For example, if $x = 2$ we have $\frac{1}{1-2}\neq 1+2+4+8+\ldots$. Explain what your instructor, the textbook and this lab mean when they say $\frac{1}{1-x}=1+x+x^2+x^3+\ldots$.

2. Find the power series representation for $f(x)=\frac{x}{9-x^2}$ by first manipulating equation (3).

3. Find the power series representation for $f(x)=\dfrac{x}{(9-x)^2}$ by first manipulating equation (3).

4. Find the power series representation for $f(x)=\ln\left(\dfrac{1+x}{1-x}\right)$ by manipulating equation (3). Hint: begin by simplifying $f(x)$ using Laws of Logs then create two power series to combine. Your final answer should be a single power series.

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5. (a) Complete the table *then find the first four terms of the Maclaurin Series* for $f(x) = \dfrac{1}{1-2x}$ using equation (7). Simplify each table entry and each coefficient of the series.

n	$f^n(x)$	$f^n(0)$
0		
1		
2		
3		

1)

(b) On the coordinate plane below, sketch the graphs of $f(x) = \dfrac{1}{1-2x}$ as well as the linear and quadratic approximations of $f(x)$ at $x = 0$. (Use your graphing calculator.)