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Indeterminate Forms and L'Hospital's Rule

Prelab: Review Lab 1, Calculating a Limit. Read L'Hospital's Rule, p. 305. Read Indeterminate Powers, p. 308, Indeterminate Differences, p. 309 and Indeterminate Powers, p. 310.

In Lab1 we explored limits and found that "direct substitution" often leads to an indeterminate form. We used a variety of algebraic methods to find these limits but that isn't always an option.

Consider $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$, where direct substitution leads to the indeterminate form $\dfrac{0}{0}$. There is no algebraic method that allows us to find this limit. A geometric method exists but it is a bit laborious (p. 191-192). This limit is evaluated quickly, using L'Hospital's Rule.

In evaluating limits, we must recognize when direct substitution leads to an indeterminate form. The following forms are indeterminate.

$$\frac{0}{0} \qquad \frac{\pm\infty}{\pm\infty} \qquad \infty - \infty \qquad 0 \cdot \pm\infty \qquad 0^0 \qquad \infty^0 \qquad 1^{\pm\infty}$$

Remember…an indeterminate form is not the same as a limit that does not exist (dne).

Example 1: Is the limit indeterminate? If so, state the form. If not, evaluate the limit or state dne.

(a) $\lim\limits_{x \to \pi} \dfrac{x - \pi}{1 - \sin x}$

(b) $\lim\limits_{x \to 0^+} x^{1/2} \ln(x)$

(c) $\lim\limits_{x \to 2^+} \dfrac{\sqrt[3]{x - 2}}{1 - \cos(x - 2)}$

(d) $\lim\limits_{x \to 0} \dfrac{e^x}{2 - x^3 + 3x^2 - 2x}$

L'Hospital's Rule:

If $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ results in either the form $\dfrac{0}{0}$ or $\dfrac{\pm\infty}{\pm\infty}$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

In words...if direct substitution leads to either of these quotient indeterminate forms, then the limit of the quotient of functions is equal to the limit of the quotient of their derivatives.
-This applies to one-sided limits also.
-If the limit still results in an indeterminate quotient, try using L'Hospital's Rule again.

*NOTE: Although we see several indeterminate forms in the box on page 9-1, the ONLY forms that allows use of L'Hospital's Rule are the quotient forms.

To evaluate the indeterminate forms:
$$\infty - \infty \qquad 0\cdot\pm\infty \qquad 0^0 \qquad \infty^0 \qquad 1^{\pm\infty}$$
Use algebraic manipulations that lead to one of the quotient indeterminate forms and then use L'Hospital's Rule.

Example 2: Complete the evaluation of the two indeterminate limits in Example 1. Indicate the use of L'Hospital's Rule.

(a) $\lim\limits_{x\to 2^+} \dfrac{\sqrt[3]{x-2}}{1-\cos(x-2)}$

(b) $\lim\limits_{x\to 0^+} x^{1/2}\ln(x)$

An indeterminate difference is usually manipulated into a quotient by finding the common denominator. However, as the next example shows, this is not always the case.

Example 3: State the indeterminate form present, then evaluate the limit, $\lim\limits_{x \to 0^+} \left(\ln x - \ln(\sin x) \right)$.

The indeterminate powers $\left(0^0, \quad \infty^0, \quad 1^{\pm\infty} \right)$ must also be manipulated into an indeterminate quotient before L'Hospital's Rule can be used. It requires a little more work. The steps are:

1. Use the natural logarithm to get the exponent "out in front" and form a product of form $0 \cdot \pm\infty$.
2. Use algebra to change this product into an indeterminate quotient.
3. Evaluate the limit using L'Hospital's Rule.
4. Remember! Step 3 is the limit of an "imposed" natural log. The final answer is "e raised to this limit".

Example 4: State the indeterminate form present, then evaluate the limit, $\lim\limits_{x \to 0} \left(2 + \dfrac{1}{x^2} \right)^{3x}$.

Math 111 F24 Lab 9 Exercises Name: _____Section: _____   Score: _____

Work each problem showing all supporting work ON THIS PAPER. You may use your textbook, lab and notes. Students may work cooperatively but each submits his/her own set of Lab Exercises. No calculator.

For each limit below: (a) *State the indeterminate form present initially and at any intermediate steps.*
(b) Evaluate the limit or determine it does not exist (DNE). Indicate when L'Hospital's Rule is used. Some problems may require use of L'Hospital's Rule more than once. Indicate any limit of $\pm\infty$.

1. $\lim\limits_{x \to 0} \dfrac{e^x - x - 1}{\cos x - 1}$

2. $\lim\limits_{x \to \infty} (1 + 3x)^{1/x}$

3. $\displaystyle\lim_{x \to 0^+}\left(\frac{1}{x} - \frac{1}{\tan x}\right)$

4. $\displaystyle\lim_{x \to \infty}\left(\frac{x}{x+1}\right)^{x}$